GLOBAL RIGIDITY OF CERTAIN ABELIAN ACTIONS BY TORAL AUTOMORPHISMS.

FEDERICO RODRIGUEZ HERTZ

Abstract. We prove global rigidity results for some linear abelian actions on tori. The type of actions we deal with includes in particular maximal rank semisimple actions on $\mathbb{T}^N$.

1. Introduction

Let Γ be a subgroup of $\mathrm{GL}(N,\mathbb{Z})$, the group of $N\times N$ matrices with integral entries and determinant ± 1. We can see Γ as acting on $\mathbb{T}^N=\mathbb{R}^N/\mathbb{Z}^N$ by matrix multiplication. In this case we will say that Γ induce a linear action or the standard action on $\mathbb{T}^N$. In general, an action of Γ on $\mathbb{T}^N$ will be an embedding $\rho:\Gamma\to \mathit{Diff}(\mathbb{T}^N)$ and we will say that ρ is an Anosov action if it has an Anosov element, i.e. if there is $m\in\Gamma$ such that $\rho(m)$ is an Anosov diffeomorphism. In this paper we will be concerned with global rigidity results for abelian linear actions on $\mathbb{T}^N$. We shall say that the standard action of Γ on $\mathbb{T}^N$ is globally rigid if any Anosov action of Γ on $\mathbb{T}^N$ which induces the standard action in homology is smoothly conjugated to it.

Theorem 1.1. *Let $A\in GL(N,\mathbb{Z})$, be a matrix with characteristic polynomial irreducible over $\mathbb{Z}$. Assume also that the centralizer $Z(A)$ of A in $GL(N,\mathbb{Z})$ has rank at least 2. Then the associated action of any finite index subgroup of $Z(A)$ on $\mathbb{T}^N$ is globally rigid.*

We want to remark that due to the Dirichlet unit theorem, in the above case, $Z(A)$ is a finite extension of $\mathbb{Z}^{r+c-1}$ where r is the number of real eigenvalues and c is the number of pairs of complex eigenvalues, $r+2c=N$. So, $Z(A)$ has rank one only if $N=2$ or if $N=3$ and A has a complex eigenvalue or if $N=4$ and A has only complex eigenvalues.

We think that the following should also be true.

Problem 1. *Let Γ be any finite index subgroup of $Z(A)$ for $A\in GL(N,\mathbb{Z})$, $N\geq 3$, assume also that $Z(A)$ is big enough. Is the standard action of Γ on $\mathbb{T}^N$ globally rigid?*

Observe that when A is the identity matrix, Γ is any finite index subgroup of $\mathrm{GL}(N,\mathbb{Z})$, see [13]. Also, one may formulate the local rigidity problem and a similar problem for actions on infra-nilmanifolds.

Date: February 2, 2008.

This work was partially supported by FCE 9021, CONICYT-PDT 29/220 and CONICYT-PDT 54/18 grants.

Classification of Anosov actions is one of the most striking problems in dynamics. When $\Gamma = \mathbb{Z}$, thus the action is generated by a diffeomorphism f, Franks [2] and Manning [17] have proven that if M is a torus, a nilmanifold or an infra-nilmanifold then f is topologically conjugated to an automorphism and thus it is essentially of an algebraic nature. In [18], Newhouse proved that codimension one Anosov diffeomorphism always live in tori. On the other hand, Brin in [1] get that with some bunching hypothesis in the spectrum of the differential of f the manifold should be an infra-nilmanifold also. It is conjectured that Anosov diffeomorphisms are always of algebraic nature, up to topological conjugacy.

When dealing with higher rank actions typically more can be said, for example that the topological conjugacy is smooth. At least this is true when ρ is a small perturbation of an irreducible algebraic Anosov action of $\mathbb{Z}^k$, $k \geq 2$, see Katok and Spatzier [15]. Moreover, it is conjectured [10], that every irreducible $\mathbb{Z}^k$, $k \geq 2$, Anosov action on any compact manifold is smoothly conjugated to an algebraic action.

Theorem 1.1 has an interesting particular case that is when dealing with Cartan actions, that is, when the matrix A has only real eigenvalues. This case was already studied by Katok and Lewis in [12] where the local rigidity property and also some global rigidity but with some restriction on the nonlinear action was established. In general one can try to push this notion of Cartan action into a broader non linear context asking the manifold M to splits into d invariant directions, $d = dim(M)$ and with the action having Anosov elements that contracts and expands this directions. Some global rigidity properties for this type of actions were studied recently by Kalinin and Spatzier in [10].

On the other hand, one can study the measure rigidity problem and in this case, already for the linear action it is not known if the unique invariant measures are Lebesgue and the atomic ones. This was first noticed by H. Furstenberg [3] who posed the problem of whether the unique invariant measures for the $\times 2$ $\times 3$ action on the circle are Lebesgue and the atomic ones. What is known is that when the entropy of the measure is positive then the measure should be Lebesgue, see [22], [6] for the $\times 2$ $\times 3$ case and [16] for the case of Cartan actions. There is also lot of work on the study of the measure rigidity for linear actions, see [7] and [8] for a good account an references about this case. But for the nonlinear case there is not much work, in [8] Kalinin and Katok remarkably have proven that $\mathbb{Z}^k$ actions on $\mathbb{T}^{k+1}$ (a priori not Anosov actions) that induce a Cartan action in homology should leave invariant a measure absolutely continuous w.r.t. Lebesgue. Later, Katok with the author in [14] proved that this measure is unique in some sense and that the action is in fact measurably isomorphic to the linear one. For the general nonlinear case, Kalinin, Katok and the author [9] prove the existence of an invariant measure absolutely continuous w.r.t. Lebesgue for quite general actions of $\mathbb{Z}^k$ on a $k+1$ dimensional manifold.

This paper grew up from a conversation with Anatole Katok during a visit of the author to the Penn State University in October 2001. At that time

he told me the problem of the global rigidity of $\mathbb{Z}^2$ actions on $\mathbb{T}^3$, this case was solved in [21]. The big step from there to here is to get rid of the one dimensionality of the invariant spaces.

I would like to thank Anatole Katok for introducing me into the wonderful subject of rigidity and also the people at Penn State for their warm hospitality.

In section 2 we shall expose some definitions that will be needed for the paper and state the main theorem. We recommend the reader to jump to the last section to see how is the scheme of the proof of the main theorem before reading sections 4 and 5. Finally section 3 is not about higher rank actions and applies to single diffeomorphisms. We think that each of sections 3, 4 and 5 have their independent interest and they are in fact quite independent.

2. Theorems and definitions

In this section we shall expose the basics notions and state the main theorem.

2.1. **Definitions.**

2.1.1. *Conjugacies.* Let $f : \mathbb{T}^N \to \mathbb{T}^N$ be an Anosov diffeomorphism and let $A \in \mathrm{GL}(N, \mathbb{Z})$ be its action in homology. By the results in [2] and [17] there is a unique conjugacy $h : \mathbb{T}^N \to \mathbb{T}^N$, $h \circ f = Ah$, homotopic to the identity, moreover, h and h^{-1} are Hölder continuous. If $\rho : \Gamma \to \mathit{Diff}(\mathbb{T}^N)$ is an abelian action with an Anosov element then the above mentioned conjugacy will conjugate the whole action, that is, if $\rho_* : \Gamma \to \mathrm{GL}(N, \mathbb{Z})$ is the induced action in homology then $h \circ \rho = \rho_* h$. So that to prove the rigidity results we shall see that h is a diffeomorphism.

Given an abelian action $\rho : \Gamma \to \mathit{Diff}(\mathbb{T}^N)$ and a point $p \in \mathbb{T}^N$, let Γ_p be the *stabilizer of* p, that is $\Gamma_p = \{n \in \Gamma \ : \ \rho(n)(p) = p\}$. When Γ_p is a finite index subgroup of Γ we say that p is a *periodic point* and call Γ/Γ_p its *period.* When ρ is an abelian Anosov action the periodic points for the action coincide with the periodic points for the Anosov element.

As we can always take a finite index subgroup of Γ isomorphic to $\mathbb{Z}^k$ and we have to prove that the conjugacy h is differentiable, we will be working typically with $\mathbb{Z}^k$ actions. Also, we will deal indistinctly with an action $\rho : \Gamma \to \mathit{Diff}(\mathbb{T}^N)$ or its image subgroup $\Gamma \sim \rho(\Gamma) \subset \mathit{Diff}(\mathbb{T}^N)$ and when working with a linear action we will simply denote $\rho_* : \Gamma \to \mathrm{GL}(d, \mathbb{R})$ or its image subgroup $\Gamma \sim \rho(\Gamma) \subset \mathrm{GL}(d, \mathbb{R})$, idem for $\mathrm{GL}(N, \mathbb{Z})$.

2.1.2. *Lyapunov exponents.* Let $\Gamma \subset \mathrm{GL}(d, \mathbb{R})$ be a subgroup isomorphic to $\mathbb{Z}^k$. Let us use the letter χ to denote the Lyapunov exponents of the action induced by $\rho : \mathbb{Z}^k \to \Gamma \subset \mathrm{GL}(d, \mathbb{R})$, hence $\chi_i(n)$ is the logarithm of the modulus of the eigenvalues of $\rho(n)$ corresponding to the *Lyapunov splitting* $\mathbb{R}^d = E_1 \oplus \cdots \oplus E_l$. We shall work with the natural extension of the Lyapunov exponents to $\mathbb{R}^k$, that is, $\chi : \mathbb{R}^k \to \mathbb{R}$ is a linear functional that coincides with the Lyapunov exponent on $\mathbb{Z}^k$. So that for every Lyapunov exponent $\chi = \chi_i$ we have the *Lyapunov space* $E_\chi = E_i$ where the eigenvalues of $\rho|E_\chi$

have modulus the exponential of χ. Given a Lyapunov space we denote χ_E the Lyapunov exponent associated with E. Given a Lyapunov space E_χ, we define the *complementary Lyapunov space* to be the invariant space $\hat{E}_\chi$ complementary to E_χ, i.e. $\hat{E}_\chi$ is the sum of all the other Lyapunov spaces, $E_\chi \oplus \hat{E}_\chi = \mathbb{R}^d$. It may be the case that two Lyapunov exponents be positively proportional, so, given a Lyapunov exponent χ, we define the *coarse Lyapunov space* $E^\chi = \bigoplus_\lambda E_\lambda$ where the sum ranges over all positive multiples $\lambda = c\chi$ of χ. We define the *complementary coarse Lyapunov space* to be the invariant space $\hat{E}^\chi$ complementary to E^χ, $E^\chi \oplus \hat{E}^\chi = \mathbb{R}^d$. So we shall also have the *coarse Lyapunov splitting* $\mathbb{R}^d = E^1 \oplus \cdots \oplus E^{l'}$ where each E^i, $1 \leq i \leq l'$ is a coarse Lyapunov space. We will call the planes $\ker \chi$ the *Weyl chamber walls* and each connected component of the complement $\mathbb{R}^k \setminus \bigcup_\chi \ker \chi$ a *Weyl chamber*. Observe that a Weyl chamber is a cone $C \subset \mathbb{R}^k$ where the Lyapunov exponents do not change sign, i.e. if $n_1, n_2 \in C \cap \mathbb{Z}^k$ then for every Lyapunov exponent χ, $\chi(n_1) > 0$ if and only if $\chi(n_2) > 0$. Given a Weyl chamber C, let us define the stable space associated to any $n \in C$, $E^s_C = \bigoplus_{\chi<0} E_\chi$, where the sum range over all Lyapunov exponents that are negative on C. Similarly we define the unstable space $E^u_C = \bigoplus_{\chi>0} E_\chi$.

See [7] for more detailed definitions.

2.2. **Main theorem.** Let $\rho_* : \mathbb{Z}^k \to \mathrm{GL}(N, \mathbb{Z})$ be an embedding and let us denote also with ρ_* the associated standard action on $\mathbb{T}^N$. We shall assume on ρ_* the following properties

i) the coarse Lyapunov splitting coincides with the splitting of $\mathbb{R}^N$ into the eigenspaces for ρ_* and the eigenvalues for ρ_* are simple, in particular the coarse Lyapunov spaces coincide with the Lyapunov spaces;
ii) on each eigenspace the set of eigenvalues form a dense subset of $\mathbb{R}^+$ or $\mathbb{C}$ depending on wether it correspond to real or complex eigenvalues;
iii) for every Weyl chamber C, and for every Lyapunov space $E \subset E^s_C$ there exists an element $m \in \mathbb{Z}^k$ with $\chi_E(m) < 0$ and $\chi_F(m) > 0$ for all other Lyapunov spaces $F \subset E^s_C$;
iv) for every Weyl chamber C we want an element $m \in \mathbb{Z}^k$ with the following bunching property $\chi^s_C(m) + \chi^{u,+}_C(m) - \chi^{u,-}_C(m) < 0$ where $\chi^s_C(m)$ is the biggest Lyapunov exponent of $\rho_*(m)|E^s_C$ and $\chi^{u,+}_C(m)$, $\chi^{u,-}_C(m)$ are the biggest and smallest Lyapunov exponents of $\rho_*(m)|E^u_C$ respectively;

Theorem 2.1. *Every linear action ρ_* on the torus with the above properties is globally rigid, that is, any Anosov action on $\mathbb{T}^N$ that induces the action ρ_* in homology is smoothly conjugated to it.*

It is not hard to see that the actions on theorem 1.1 satisfy hypothesis i)–iv). Besides, if for example $\rho_1 : \mathbb{Z}^{k_1} \to \mathrm{GL}(N_1, \mathbb{Z})$ and $\rho_2 : \mathbb{Z}^{k_2} \to \mathrm{GL}(N_2, \mathbb{Z})$ are actions as in theorem 1.1 then the product action $\rho_* : \mathbb{Z}^{k_1+k_2} \to \mathrm{GL}(N_1 + N_2, \mathbb{Z})$ given by $\rho_*(n_1, n_2) = (\rho_1(n_1), \rho_2(n_2))$ also satisfies these hypothesis and hence we can apply theorem 2.1 and this product action is globally rigid.

We think that in fact if any two actions satisfy hypothesis i)–iv) then their product should also satisfy these hypothesis, the first 3 are easily seen, but we were not able to see how to get hypothesis iv).

Other types of actions satisfying the above hypothesis are the following. Recall that $\mathrm{Sp}(n,\mathbb{Z})$ is the group of symplectic $n\times n$ matrices with integral entries (clearly n is even).

Theorem 2.2. *Let $A \in Sp(N,\mathbb{Z})$, $N \geq 4$ be a matrix with characteristic polynomial irreducible over $\mathbb{Z}$, if $N = 4$ assume also that A has at least one real eigenvalue. Then the standard action associated to any finite index subgroup of $Z(A) \cap Sp(N,\mathbb{Z})$ is globally rigid.*

Finally, the smoothness required in theorem 2.1 is C^2 although a $C^{1+\alpha}$ hypothesis would be enough, for some $0 < \alpha < 1$ that a priori may depend on the action ρ_*.

3. Smoothness of holonomies

Proposition 3.1. *Let $f, g : M \to M$ be $C^{1+\text{Hölder}}$ diffeomorphisms. Let μ be an invariant measure for f and assume there is a Hölder continuous homeomorphism $h : U \to V$ from a neighborhood U of the support of μ onto $V \subset N$ such that $h \circ f = g \circ h$. Let us call $\nu = h_*\mu$. Then, for $\mu-$a.e. x, the number of negative Lyapunov exponents at x are less than or equal to the ones at $h(x)$.*

In the proof we shall use the strong stable (unstable) manifold theorem, see for instance [19]

Theorem 3.2. Pesin strong stable manifold theorem. *Let $f : M \to M$ be a $C^{1+\text{Hölder}}$ diffeomorphisms. Let μ be an invariant measure for f. There is a set of full μ-measure R_μ, the $\mu-$regular points, such that if $x \in R_\mu$ and $T_xM = E_1(x) \oplus E_2(x)$ where the Lyapunov exponents corresponding to $E_1(x)$ are negative and less than the Lyapunov exponents corresponding to $E_2(x)$, then there is a unique manifold $W_{E_1}(x)$ tangent to $E_1(x)$ at x and characterized as the points y such that $d(y,x) \leq \varepsilon(x)$ for some $\varepsilon(x) > 0$ and*

$$\limsup_{n\to+\infty} \frac{1}{n} \log d\left(f^n(y), f^n(x)\right) < \inf\{\chi_2(x), 0\}$$

where $\chi_2(x)$ is the smallest Lyapunov exponent corresponding to $E_2(x)$.

Proof of proposition 3.1. Take a $\mu-$regular point x and assume that $h(x)$ is also an $\nu-$regular point (as $\nu = h_*\mu$, this holds for $\mu-$a.e. point). Take the splitting $T_xM = E^s(x) \oplus E^c(x) \oplus E^u(x)$ w.r.t. negative, zero and positive Lyapunov exponents and let $W^s(x)$ be the invariant manifold tangent to $E^s(x)$ given by the Pesin strong stable manifold theorem. Do the corresponding counterpart at $h(x)$. As h is Hölder continuous, $h(W^s(x)) \subset W^s(h(x))$. Then, using the invariance of domain theorem we get that $\dim(W^s(x)) \leq \dim(W^s(h(x)))$ and we are done. $\square$

Corollary 3.3. *In the setting of proposition 3.1, if μ is a hyperbolic measure, then ν is also a hyperbolic measure with $\dim E^s(x) = \dim E^s(h(x))$ and $\dim E^u(x) = \dim E^u(h(x))$ $\mu-$a.e. x.*

Remark 1. *If μ is a hyperbolic measure, it can also be proved that if $\chi_\mu^- \leq \log\lambda < 0$ (resp. $\chi_\mu^+ \geq \log\sigma > 0$) for every negative (resp. positive) Lyapunov exponent, then $\chi_\nu^- \leq \theta\log\lambda$ (resp. $\chi_\nu^+ \geq \theta\log\sigma > 0$) for every negative (resp. positive) Lyapunov exponent, where θ is a Hölder exponent for h.*

An improved version of the next proposition, where the existence of the sequence b_n is not needed, already appeared in [Sc], we include a proof here because it is simpler in our case.

Proposition 3.4. *Let $f : X \to X$ be a continuous map of a compact metric space. Let $a_n : X \to \mathbb{R}$, $n \geq 0$ be a sequence of continuous functions such that $a_{n+k}(x) \leq a_n(f^k(x)) + a_k(x)$ for every $x \in X$, $n, k \geq 0$ and such that there is a sequence of continuous functions b_n, $n \geq 0$ satisfying $a_n(x) \leq a_n(f^k(x)) + a_k(x) + b_k(f^n(x))$ for every $x \in X$, $n, k \geq 0$. If*

$$\inf_n \frac{1}{n}\int_X a_n d\mu < 0$$

for every ergodic f-invariant measure, then there is $N \geq 0$ such that $a_N(x) < 0$ for every $x \in X$.

Proof. For an invariant measure μ, let us call $a_n(\mu) = \int_X a_n d\mu$. We have that $a_{n+k}(\mu) \leq a_n(\mu) + a_k(\mu)$. Now, if $\inf_n \frac{a_n(\mu)}{n} < 0$ for every ergodic f-invariant measure, then the same holds for every invariant measure by the ergodic decomposition theorem and the multiplicative ergodic theorem. Because of the compactness of the set of invariant measures and the properties of the $a_n(\mu)$, there is $m \geq 0$ such that $a_m(\mu) < c < 0$ for every invariant measure μ. This implies that for some $n_0 > 0$

$$\sum_{j=0}^{n-1} a_m(f^j(x)) < cn$$

for every $x \in X$, $n \geq n_0$. Take $N = lm$ for some $l \geq 0$ big enough, then

$$\sum_{h=0}^{m-1}\sum_{i=0}^{l-1} a_m(f^{im}(f^h(x))) = \sum_{j=0}^{N-1} a_m(f^j(x)) < cN$$

Thus, we have, by the properties of the a_n's, that

$$\sum_{h=0}^{m-1} a_N(f^h(x)) < cN$$

and hence

$$\begin{aligned} ma_N(x) &\leq \sum_{h=0}^{m-1} a_N(f^h(x)) + a_h(x) + b_h(f^N(x)) \\ &\leq cN + m(\sup_{x\in X;\, h<m} a_h + \sup_{x\in X;\, h<m} b_h) \end{aligned}$$

Finally, taking l big enough, as $c < 0$ we get the proposition. □

Applying proposition 3.4 to the functions $a_n(x) = \log |D_x f^n|E|$ and $b_n(x) = \log m(D_x f^n|E)$ we get the following immediate corollaries of the above proposition. A regular C^1 map is a map whose derivative is invertible at each point.

Corollary 3.5. *Let $f : M \to M$ be a regular C^1 map and Λ a compact invariant set. Assume f leaves invariant a continuous bundle E over Λ. If the Lyapunov exponents of the restriction of Df to E are all negative (positive) for every ergodic invariant measure, then Df contracts (expands) E uniformly.*

Corollary 3.5 already appeared in [Ca]. The following is a corollary of the above and Corollary 3.3.

Corollary 3.6. *Let $f : M \to M$ be a diffeomorphism and $g : N \to N$ be a $C^{1+Hölder}$ diffeomorphism. Let Λ be a transitive hyperbolic set for f and assume there is a Hölder continuous homeomorphism $h : U \to V$ from a neighborhood U of Λ onto $V \subset N$ such that $h \circ f = g \circ h$. Let us assume that g leaves a continuous invariant splitting $TM = E_1 \oplus E_2$ over $h(\Lambda) = \Lambda_g$ and that it coincides with the Lyapunov (stable$\oplus$unstable) splitting for some (necessarily) hyperbolic g-invariant measure. Then Λ_g is a hyperbolic set for g.*

Proof. Although in Corollary 3.3 f is assumed to be $C^{1+\text{Hölder}}$, it is not hard to see that Λ being a hyperbolic set, this hypothesis can be removed. □

Similarly we have the following corollary that states that the fact of being an expanding map is preserved by Hölder conjugacies.

Corollary 3.7. *Let $f : M \to M$ be an expanding map and $g : N \to N$ be a $C^{1+Hölder}$ regular map. Assume there is a Hölder continuous homeomorphism $h : M \to N$ conjugating f and g, i.e. $h \circ f = g \circ h$. Then g is an expanding map.*

The following has been recently proven by Wenxiang Sun and Zhenqi Wang after the work of Anatole Katok [11]

Theorem 3.8. [23] *Let $g : M \to M$ be a $C^{1+\alpha}$ diffeomorphism and let μ be an ergodic hyperbolic measure. Then the Lyapunov exponents of μ can be approximated by the Lyapunov exponents of periodic orbits.*

The following theorem is essentially proved in [5] and [20]. I would like to thank Keith Burns for pointing out this theorem to me.

Theorem 3.9. *Let $f : M \to M$ be a C^k diffeomorphism with an invariant splitting $TM = E_1 \oplus E_2$ satisfying $\sup_p \frac{\|D_pf|_{E_1}\|}{m(D_pf|_{E_2})} < 1$. Let us assume that*

$$\sup_x \|D_xf|_{E_1}\| \frac{\|D_xf|_{E_2}\|^r}{m(D_xf|_{E_2})} < 1$$

Then there is a C^s foliation tangent to E_1 where $s = \min\{k-1, r\}$.

Finally we have the following corollary that get smoothness of the strong stable foliation for an Anosov diffeomorphism form periodic data.

Corollary 3.10. *Let g be a C^k Anosov diffeomorphism, and assume it preserves a continuous splitting $TM = E_1 \oplus E_2$ (not necessarily the hyperbolic splitting). Given a periodic point p, let us call $\chi_1^+(p)$ the biggest Lyapunov exponent of the restriction of Df to E_1, $\chi_2^+(p)$ the biggest Lyapunov exponent of the restriction of Df to E_2 and $\chi_2^-(p)$ the smallest Lyapunov exponent of the restriction of Df to E_2. If there is a constant $c < 0$ such that $\chi_1^+(p) - \chi_2^-(p) < c < 0$ and $\chi_1^+(p) + r\chi_2^+(p) - \chi_2^-(p) < c < 0$, where $r \geq 1$, for every periodic point p then there is a C^s foliation tangent to E_1 where $s = \min\{k-1, r\}$.*

4. Smooth linearization in $\mathbb{R}^d$.

In this section we shall prove a result about smooth linearization of some abelian actions in $\mathbb{R}^d$ that fix the origin. We shall follow the proof of Hartman in [4] of smooth linearization of contractions.

Take $\rho : \mathbb{Z}^k \to \mathit{Diff}^{\,2}(\mathbb{R}^d, 0)$ an action fixing the origin. Let us assume that there is $n_0 \in \mathbb{Z}^k$ such that $D_0\rho(n_0)$ is a contraction. Write $T = \rho(n_0)$ then $\mathbb{R}^d$ splits as a direct D_0T-invariant sum $\mathbb{R}^d = E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_n}$ where $0 < \lambda_1 < \cdots < \lambda_n < 1$ and the eigenvalues of $D_0T|E_{\lambda_i}$ have modulus λ_i. We shall assume that for every $i = 1, \ldots, n$ there is a C^2 manifold, W^i tangent to $E_{\lambda_i} \oplus \cdots \oplus E_{\lambda_n}$ and invariant by ρ in the following sense: for every $n \in \mathbb{Z}^k$, there is $\varepsilon > 0$ such that $\rho(n)(W^i \cap B_\varepsilon(0)) \subset W^i$.

Theorem 4.1. *Let $\rho : \mathbb{Z}^k \to \mathit{Diff}^{\,2}(\mathbb{R}^d, 0)$ be an action as above, then there is a $C^{1+H\ddot{o}lder}$ diffeomorphism h such that $h \circ \rho = D_0\rho \circ h$.*

As in [4], the C^2 condition can be relaxed to a $C^{1+\alpha}$ hypothesis for some $0 < \alpha < 1$ that depends on the eigenvalues of the action $D_0\rho$. Let us point out that the existence of the manifolds W^i is non trivial at all. The following is an example where these manifolds are not present: $f(x,y) = (\lambda^2 x, \lambda y)$ and $g(x,y) = (x + y^2, y)$, $\lambda < 1$ commute, f is a linear contraction, but this $\mathbb{Z}^2$-action is not linearizable, and there is no invariant manifold W^2 tangent to the vertical direction.

Theorem 4.1 is of a local nature, in fact the action needs only to be a germ of action and there will be also a smooth local linearization. Moreover, once there is a local linearization, it can be extended globally using the contraction $T = \rho(n_0)$ in the obvious manner.

Proof. We shall show here how to adapt the proof in [4] to our case. So let us describe how this proof works. Hartman's proof is essentially by induction, he assumed the coordinates associated to $E_{\lambda_{i+1}} \oplus \cdots \oplus E_{\lambda_n}$ are already linearized, then he found an invariant manifold tangent to $E_{\lambda_{i+1}} \oplus \cdots \oplus E_{\lambda_n}$, he make a first conjugacy sending this invariant manifold into $E_{\lambda_{i+1}} \oplus \cdots \oplus E_{\lambda_n}$ and finally he linearize the E_{λ_i} coordinate, without touching the already linearized coordinates. Finally, the first induction step is trivial by adding some dummy coordinates.

In our case, let us make first a smooth conjugacy and assume that the manifolds W^i are already the spaces $E_{\lambda_i} \oplus \cdots \oplus E_{\lambda_n}$ and hence that the action ρ preserves this spaces.

Then, we follow the proof of Hartman. Write an N-vector as (x, y, z) where x is an I-vector, y a J-vector and z a K-vector and $I+J+K = d$. Let A, B, C be square matrices of order I, J, K and with eigenvalues $a_1, \dots, a_I$, $b_1, \dots, b_J$, $c_1, \dots, c_K$ respectively.

Induction hypothesis. Assume that T is written as

$$T : x^1 = Ax + X(x, y, z), \quad y^1 = By + Y(x, y, z), \quad z^1 = Cz,$$

where the eigenvalues of A, B, C, satisfy

$$0 < |a_1| \leq \cdots \leq |a_I| < |b_1| = \cdots = |b_J| < |c_1| \leq \cdots \leq |c_K| < 1;$$

and X, Y satisfy

(1) X, Y are C^1 and $|(X, Y)(x, y, z)| \leq L\,(|x| + |y| + |z|)\,(|x| + |y|)$;
(2) $\partial_x X, \partial_y X$ and $\partial_x Y, \partial_y Y$ are uniformly Lipschitz continuous w.r.t. (x, y, z);
(3) $\partial_z X, \partial_z Y$ are uniformly Lipschitz continuous w.r.t. (x, y);
(4) $\partial_z X, \partial_z Y$ are uniformly Hölder continuous w.r.t. z.

Then, as in Hartman's theorem, theorem 4.1 is proven if the following is verified.

Induction assertion. There exists a map R of the form

$$R : u = x, \quad v = y - \varphi(x, y, z), \quad w = z,$$

where φ satisfy

a) φ is C^1 and $|\varphi(x, y, z)| \leq L\,(|x| + |y| + |z|)\,(|x| + |y|)$;
b) $\partial_x \varphi, \partial_y \varphi$ are uniformly Lipschitz continuous w.r.t. (x, y, z);
c) $\partial_z \varphi$ is uniformly Lipschitz continuous w.r.t. (x, y);
d) $\partial_z \varphi$ is uniformly Hölder continuous w.r.t. z.

R is such that $F = R \circ T \circ R^{-1}$ has the form

$$F : u^1 = Au + U(u, v, w), \quad v^1 = Bv, \quad w^1 = Cw,$$

where

(1) U is C^1 and $|U(u, v, w)| \leq L(|u| + |v| + |w|)|u|$;
(2) $\partial_u U$ is uniformly Lipschitz continuous w.r.t. (u, v, w);
(3) $\partial_v U, \partial_w U$ are uniformly Lipschitz continuous w.r.t. u;
(4) $\partial_v U, \partial_w U$ are uniformly Hölder continuous w.r.t. (v, w).

We put one more assertion that is

(5) If G has the form

$$G : u^1 = A_G u + U_G(u,v,w), \quad v^1 = B_G v + V_G(u,v,w), \quad w^1 = C_G w,$$

where $|V_G(u,v,w)| \leq L(|u| + |v| + |w|)(|u| + |v|)$ and G commutes with $F = R \circ T \circ R^{-1}$ then $V_G \equiv 0$.

Observe that a map $\eta(a,b)$ satisfies that $|\eta(a,b)| \leq L(|a| + |b|)|a|$ if η is C^1, $\partial_a \eta$ is uniformly Lipschitz continuous w.r.t. (a,b), $\partial_a \eta(0,0) = 0$ and $\eta(0,b) = 0$.

The construction of the conjugacy R and the proof of assertions (1)–(4) follows exactly the lines in [4]. Let us see the proof of assertion (5).

We have that $F \circ G = G \circ F$ implies that $V_G \circ F^n = B^n V_G$. Let us write the first component in F^n as $(F^n)_1$, that is

$$F^n(u,v,w) = \left((F^n)_1(u,v,w), B^n v, C^n w\right).$$

Since $|U(u,v,w)| \leq L(|u| + |v| + |w|)|u|$ we have that for any $\lambda > |a_K|$, $\lambda^{-n}|(F^n)_1(u,v,w)| \to 0$ if (u,v,w) is close enough to 0. So we have that

$$\begin{aligned} |B^n V_G(u,v,w)| &= |V_G(F^n(u,v,w))| \\ &\leq L(|(F^n)_1| + |B^n v| + |C^n w|)(|(F^n)_1| + |B^n v|) \end{aligned}$$

where the argument of $(F^n)_1$ is (u,v,w). Then, taking $|c_K| < \mu < 1$ we have that

$$|(F^n)_1| + |B^n v| + |C^n w| \leq C\mu^n$$

and then

$$||b_1|^{-n} B^n V_G(u,v,w)| \leq LC\mu^n(|b_1|^{-n}|(F^n)_1| + |b_1|^{-n}|B^n v|)$$

Since the matrix $|b_1|^{-1}B$ has all its eigenvalues of modulus one, we get that the norm of the matrix $|b_1|^{-n}B^n$ is bounded between Cn^J and $C^{-1}n^{-J}$ and hence

$$C^{-1}n^{-J}|V_G(u,v,w)| \leq LC\mu^n(C + Cn^J)$$

which gives that $V_G(u,v,w) = 0$ if (u,v,w) is close to 0, then using that F is a contraction we can dispense the requirement (u,v,w) is close to 0.

So that assertion (5) says that when we linearize $T = \rho(n_0)$ we also linearize the whole action. Indeed, the elements of our action satisfy the requirement on V_G since they preserve the spaces $E_{\lambda_i} \oplus \cdots \oplus E_{\lambda_n}$ and also the conjugacy R preserves that spaces.

□

To apply the theorem above we will need the following proposition

Lemma 4.2. *Let $\rho_* : \mathbb{Z}^k \to \mathbb{C} \setminus \{0\}$ be a linear action induced by complex multiplication and let $\rho : \mathbb{Z}^k \to$ Diff$(\mathbb{C}, 0)$ be another action. Assume that there is a homeomorphism $h : \mathbb{C} \to \mathbb{C}$ such that $h \circ \rho = \rho_* \circ h$. Assume also that the image of ρ_* is dense in $\mathbb{C}$. Then, $D_0\rho(n)$ has complex eigenvalues for every n.*

Proof. If for some n_0, $D_0\rho(n_0)$ has its eigenvalues of the same modulus, then applying theorem 4.1 and lemma 5.8 we get the desired result. So let us assume that $D_0\rho(n_0)$ has two eigenvalues. Then by the strong stable manifold theorem, there is a unique $\rho(n_0)$-invariant manifold tangent to the eigenspace of the smallest eigenvalue. Then, by uniqueness, this manifold should be invariant by the whole action. On the other hand, it should be dense by hypothesis, this gives a contradiction. $\square$

5. Rigidity for Hölder conjugacies at periodic orbits.

In this section we shall prove that if a linear action in $\mathbb{R}^d$ is Hölder conjugated to a sufficiently rich linear action then the conjugacy should split.

Definition 5.1. *We say that the action $\rho_* : \mathbb{Z}^k \to GL(d,\mathbb{R})$ is* rich *if there is an element $\rho_*(n_0)$ that is a contraction and for every coarse Lyapunov space E_i^*, $i = 1, \dots l$ there is n_i such that, $\rho_*(n_i)$ is a contraction when restricted to E_i^* and an expansion on the complement.*

Theorem 5.2. *Let $\rho_*, \rho : \mathbb{Z}^k \to GL(d,\mathbb{R})$ be linear actions on $\mathbb{R}^d$. Assume ρ_* is a rich action and that $h \circ \rho(n) = \rho_*(n) \circ h$ for every $n \in \mathbb{Z}^k$, where $h : \mathbb{R}^d \to \mathbb{R}^d$ is a homeomorphism Hölder continuous in a neighborhood of the origin. Then h is of the form*

$$h(x_1, \dots, x_l) = \big(h_1(x_1), \dots, h_l(x_l)\big)$$

where $x = (x_1, \dots, x_l)$ is taken w.r.t. the coarse Lyapunov splitting for ρ and $h(x) = \big(h_1(x_1), \dots, h_l(x_l)\big)$ is taken w.r.t. the coarse Lyapunov splitting for ρ_*.

Observe that it is an implicit consequence of the theorem the fact that ρ will have a splitting $E_1 \oplus \cdots \oplus E_l$ that will coincide with the coarse Lyapunov splitting.

Lemma 5.3. *Let $\rho_* : \mathbb{Z}^k \to GL(d,\mathbb{R})$ be an action such that all its Lyapunov exponents are positively proportional. Let $\rho : \mathbb{Z}^k \to GL(d,\mathbb{R})$ be another action and assume that there is an homeomorphism $h : \mathbb{R}^d \to \mathbb{R}^d$ such that $h \circ \rho = \rho_* \circ h$. Then, the Lyapunov exponents of ρ are positively proportional to the ones of ρ_*.*

Proof. Since h conjugates ρ and ρ_* and both are linear, for every $n \in \mathbb{Z}^k$, $\chi_i(n) < 0$ if and only if $\chi(n) < 0$. Since χ_i and χ are linear functionals the result follows. $\square$

Let $\rho_* : \mathbb{Z}^k \to \mathrm{GL}(d,\mathbb{R})$ be an action, E_1 a coarse Lyapunov space and E_2 the complementary coarse Lyapunov space, $\mathbb{R}^d = E_1 \oplus E_2$. Assume that there is $n_0 \in \mathbb{Z}^k$ such that $\rho_*(n_0)$ is a contraction and that there is $n_1 \in \mathbb{Z}^k$ such that $\rho_*(n_1)$ is an expansion on E_1 and a contraction on E_2.

Proposition 5.4. *Let $\rho_* : \mathbb{Z}^k \to GL(d,\mathbb{R})$ be as above and let $\rho : \mathbb{Z}^k \to GL(d,\mathbb{R})$ be another action. Assume that there is $h : \mathbb{R}^d \to \mathbb{R}^d$ a homeomorphism that is Hölder continuous in a neighborhood of the origin and such that*

$h\circ\rho=\rho_*\circ h$. Then we have that $h^{-1}(E_i)$, $i=1,2$, are complementary linear subspaces, preserved by ρ and $h_1(x_1,x_2)=h_1(x_1,0)$ where $x=(x_1,x_2)$ are coordinates with respect to $h^{-1}(E_1)\oplus h^{-1}(E_2)$ and $h=(h_1,h_2)$ are coordinates with respect to $E_1\oplus E_2$

Notice that in this proposition the inverse of h is not required to be Hölder continuous at all. Moreover, it seems that the hypothesis of being a homeomorphisms could be relaxed.

To proof the proposition we shall use the following lemma

Lemma 5.5. *Let $C,\bar C\in GL(d,\mathbb{R})$ leave invariant a splitting $E_1\oplus E_2$. Assume that C and $\bar C$ are contractions and that there is $h:\mathbb{R}^d\to\mathbb{R}^d$ a map that is Hölder continuous in a neighborhood of the origin with Hölder exponent β such that $h\circ\bar C=C\circ h$. Let us call χ_1 the smallest Lyapunov exponent of $C|_{E_1}=C_1$ and $\bar\chi_2$ the biggest Lyapunov exponent of $\bar C|_{E_2}=\bar C_2$. If $\beta\bar\chi_2<\chi_1$ then $h_1(x_1,x_2)=h_1(x_1,0)$ where h_1 is the component of h in E_1.*

Proof. Denote $\bar C_1=\bar C|_{E_1}$. We may assume without loss of generality that the neighborhood where h is Hölder is the unit ball (in fact by the conjugacy property, h is Hölder in all $\mathbb{R}^d$). Take $x=(x_1,x_2)$ and take $n\geq 0$ big enough such that $|\bar C^n x|\leq 1$, then

$$\begin{aligned}\big|h_1(x_1,x_2)-h_1(x_1,0)\big| &= \Big|C_1^{-n}\Big[h_1\big(\bar C_1^n x_1,\bar C_2^n x_2\big)-h_1\big(\bar C_1^n x_1,0\big)\Big]\Big|\\ &\leq K\big\|C_1^{-n}\big\|\Big|\bar C_2^n x_2\Big|^\beta\leq K\big\|C_1^{-n}\big\|\big\|\bar C_2^n\big\|^\beta|x_2|^\beta\end{aligned}$$

where $K>0$ is a generic constant. The last expression tends to 0 as $n\to+\infty$. Indeed, take $\varepsilon>0$ small such that still $V=(1-\varepsilon)\beta\bar\chi_2-(1+\varepsilon)\chi_1<0$. If n is big enough, then

$$\big\|C_1^{-n}\big\|\leq exp\big(n\big[-\chi_1(1+\varepsilon)\big]\big)$$

and also

$$\big\|\bar C_2^n\big\|\leq exp\big(n\big[\bar\chi_2(1-\varepsilon)\big]\big)$$

so that

$$\big\|C_1^{-n}\big\|\big\|\bar C_2^n\big\|^\beta\leq\exp\big(n\big[\beta\bar\chi_2(1-\varepsilon)-\chi_1(1+\varepsilon)\big]\big)=\exp(nV)$$

and we are done. □

Proof of proposition 5.4. Let as assume that the Hölder exponent of h is β. Take $A=\rho_*(n_0)$ and $B=\rho_*(n_1)$ and denote $A_i=A|_{E_i}$ and $B_i=B|_{E_i}$, $i=1,2$. As B_1 is an expansion and B_2 a contraction, h is a homeomorphism and ρ is linear, by the stable manifold theorem and the uniqueness of stable and unstable manifolds we get that $h^{-1}(E_i)$, $i=1,2$ should be linear ρ-invariant subspaces. So we may assume, without loss of generality that ρ already leaves invariant the splitting $E_1\oplus E_2$. Let us denote also $\bar A=\rho(n_0)$, $\bar B=\rho(n_1)$, $\bar A_i=\bar A|_{E_i}$ and $\bar B_i=\bar B|_{E_i}$, $i=1,2$. As h is a homeomorphism we have that $\bar A_2$ and $\bar B_2$ are contractions and hence there is $a>0$ such that if $\bar\chi_2(l,m)$ is the biggest Lyapunov exponent of $\bar A_2^l\bar B_2^m$ then $\bar\chi_2(l,m)\leq -a(l+m)$ for $l,m\geq 0$. Call χ_1 the Lyapunov exponent for $\rho_*|E_1$ such that all the other Lyapunov

exponents when restricted to E_1 has rate of proportionality less that 1. Call $\chi_1^A = \chi_1(n_0) < 0$ and $\chi_1^B = \chi_1(n_0) > 0$ then the Lyapunov exponent χ_1 of $A_1^l B_1^m$ is $\chi_1(l,m) = l\chi_1^A + m\chi_1^B$. For any $l > 0$, there is $m_l \geq 0$ such that

$$-1 \leq l\frac{\chi_1^A}{\chi_1^B} + m_l < 0$$

So $-\chi_1^B \leq l\chi_1^A + m_l\chi_1^B = \chi_1(l,m_l) < 0$ and hence, as all the other Lyapunov exponents of $\rho_*|E_1$ are positively proportional we have that $C = A^l B^{m_l}$ is a contraction and by the choice of χ_1, that it is the smallest Lyapunov exponent of $C|_{E_1}$. Hence we get that

$$\beta\bar{\chi}_2 - \chi_1 = \beta\bar{\chi}_2(l,m_l) - \chi_1(l,m_l) \leq -\beta a(l+m_l) + \chi_1^B$$

Taking l big enough the right hand side is negative. By lemma 5.5 we get the desired property. □

The proof of theorem 5.2 is an immediate application of proposition 5.4.

Lemma 5.6. *Let $\rho_*, \rho : \mathbb{Z}^k \to \mathbb{R}^+$ be linear actions on the line, and assume that there is a continuous non constant map $h : \mathbb{R}^+ \to \mathbb{R}^+$, continuous at 0, such that $h \circ \rho = \rho_* \circ h$. Assume also that the image of ρ_* is dense in $\mathbb{R}^+$. Then, there are $t > 0$ and $\alpha \in \mathbb{R}^+$ such that $h(x) = \alpha x^t$ and $\rho_* = \rho^t$. Moreover, if h is absolutely continuous with non-zero jacobian at 0 then $t = 1$ and hence $\rho_* = \rho$.*

Proof. First of all, either the image of ρ is dense or discrete. If it where discrete, then we will have a vector $v \in \mathbb{Z}^N$ and $\lambda > 0$ such that $\rho(n) = \lambda^{v\cdot n}$. On the other hnd, as the image of ρ_* is dense, there should be $n \in \mathbb{Z}^N$ such that $\rho_*(n) \neq 1$ and $v \cdot n = 0$. Hence we get that for any x, and such n, $h(x) = h(\rho(n)x) = \rho_*(n)h(x)$ which is possible only if $h(x) = 0$ and hence h is trivial in which case the proposition is trivial also. Let us assume that the image of ρ is also dense. Then we have that $h(x) > 0$ for every $x \neq 0$. Hence, if $\rho_*(n) = 1$ then $\rho(n) = 1$, if $\rho_*(n) > 1$ then $\rho(n) > 1$ and if $\rho_*(n) < 1$ then $\rho(n) < 1$. This is only possible if there is t positive such that $\rho_* = \rho^t$. Hence, if we put $\alpha = h(1)$ then we get the result. □

An immediate corollary is the following.

Corollary 5.7. *Let $\rho_*, \rho : \mathbb{Z}^k \to \mathbb{R}^+$ be linear actions on the line, and assume that there is a continuous map $h : \mathbb{R} \to \mathbb{R}$ such that $h \circ \rho = \rho_* \circ h$. Assume also that the image of ρ_* is dense in $\mathbb{R}^+$. Then, there are $t \geq 0$ and $\alpha_\pm \in \mathbb{R}$ such that $h(x) = \alpha_\pm |x|^t$ for $x \in \mathbb{R}^\pm$. Moreover, if h is not trivial then $t \neq 0$, if h is absolutely continuous with non-zero jacobian at 0 then $t = 1$ and hence $\rho_* = \rho$ and if the jacobian is continuous at 0 then $\alpha_+ = -\alpha_- = \alpha$ and if h preserves orientation then $\alpha > 0$.*

Lemma 5.8. *Let $\rho_* : \mathbb{Z}^k \to \mathbb{C} \setminus \{0\}$ be a linear action induced by complex multiplication and let $\rho : \mathbb{Z}^k \to GL(2,\mathbb{R})$ be another action. Assume that there is a homeomorphism $h : \mathbb{C} \to \mathbb{C}$ such that $h \circ \rho = \rho_* \circ h$. Assume*

also that the image of ρ_* is dense in $\mathbb{C}$. Then, after a linear conjugacy, ρ is induced by complex multiplication. Moreover, there are $t > 0$, $\alpha \in \mathbb{C} \setminus \{0\}$ and $a \in \mathbb{R}$ such that if h preserves orientation, then $h(z) = \alpha z|z|^{t-1} \exp(ia \log |z|)$ and if h reverses orientation, then $h(z) = \alpha \bar{z}|z|^{t-1} \exp(ia \log |z|)$. Moreover, if h is absolutely continuous at 0 with non-zero jacobian then $t = 1$ and hence $|\rho_*| = |\rho|$ and if h is differentiable at 0, then $a = 0$ and either h preserves orientation and $\rho_* = \rho$ or h reverses orientation and $\rho_* = \bar{\rho}$.

Proof. Let us prove first that ρ is induced by complex multiplication. Assume by contradiction that ρ leaves invariant the line $xe^{i\theta}$, $x \in \mathbb{R}$. Then, using corollary 5.7 we have that $|h(xe^{i\theta})| = \alpha x^t$ for some $\alpha > 0$ and for every $x > 0$. On the other hand, as the image of ρ_* is dense, we have that the image of $xe^{i\theta}$, $x > 0$ by h must be dense in $\mathbb{C}$. But if we take x_n such that $h(x_n e^{i\theta}) \to z$, then this implies that $x_n \to \frac{|z|^{1/t}}{\alpha}$ but then it would be impossible to approach any other point with the same modulus of z. So that we may assume that ρ is induced by complex multiplication.

As h is a homeomorphism we have that the image of ρ is dense in $\mathbb{C}$. Dividing by $h(1)$ if necessary we may assume that $h(1) = 1$. Recall that $\mathbb{C}$ is the universal cover of $\mathbb{C} \setminus \{0\}$ with the exponential being the covering map. Thus, we may take a lift H of h such that $h(e^z) = e^{H(z)}$, $H(z + 2\pi i) = H(z) + 2\pi i$ and lifts $\hat{\rho}_*$ and $\hat{\rho}$ of ρ_* and ρ respectively in such a way that $\hat{\rho}_*, \hat{\rho} : \mathbb{Z}^k \to \mathbb{C}$ be homomorphisms acting on $\mathbb{C}$ by translation, $e^{\hat{\rho}_*} = \rho_*$ and $e^{\hat{\rho}} = \rho$. Hence $H(z + \hat{\rho}(n)) = H(z) + \hat{\rho}_*(n)$. Thus our hypothesis gives us that $\{\hat{\rho}_*(n) + l2\pi i : n \in \mathbb{Z}^k; l \in \mathbb{Z}\}$ and $\{\hat{\rho}(n) + l2\pi i : n \in \mathbb{Z}^k; l \in \mathbb{Z}\}$ are dense. But then H should be affine, because we may see H as a conjugacy between actions by dense translations on tori. Once H is affine, as $H(0) = 0$ we have that H is linear and as $H(2\pi i) = 2\pi i$ we get that $H(x + yi) = tx + (ax + y)i$ for some real numbers $t \neq 0$ and a. Thus we get the lemma.

$\square$

It seems likely that the hypothesis of h being an homeomorphisms could be relaxes.

6. Putting all together.

First let us put a corollary of sections 4 and 5. Let $\rho_* : \mathbb{Z}^k \to \mathrm{GL}(d, \mathbb{R})$ be a linear action by semi-simple matrices. Assume that the coarse Lyapunov splitting coincides with the splitting into eigenspaces so that each coarse Lyapunov space has dimension one or two depending on wether it corresponds to a real eigenvalue or a complex eigenvalue. Let us assume also that ρ_* is a rich action as in definition 5.1 and that on each Lyapunov direction the set of eigenvalues form a dense subset of $\mathbb{R}^+$ or $\mathbb{C}$ depending on wether it correspond to real or complex eigenvaules.

Theorem 6.1. *Let $\rho_* : \mathbb{Z}^k \to GL(d, \mathbb{R})$ be an action as above and let $\rho : \mathbb{Z}^k \to \mathit{Diff}^{\,2}(\mathbb{R}^d, 0)$ be an action fixing the origin. Assume there is a conjugacy $h : \mathbb{R}^d \to \mathbb{R}^d$, $h \circ \rho = \rho_* \circ h$ that is Hölder continuous in a neighborhood of*

0*. Then h is a diffeomorphism outside the preimage by h of the union of the ρ_*-complementary coarse Lyapunov spaces. Moreover,*

a) the corresponding Lyapunov exponents for ρ and for ρ_* are proportional;
b) h is absolutely continuous at 0 with nonzero jacobian if and only if the corresponding Lyapunov exponents for ρ and for ρ_* coincide and in this case h is absolutely continuous with nonzero jacobian at every point;
c) h is differentiable at 0 with nonzero jacobian if and only if the corresponding eigenvalues for ρ and for ρ_* coincide and in this case h is a diffeomorphism.

Proof. From the rich property for ρ_*, the existence of the conjugacy h and lemma 4.2 it follows that the action ρ is in the hypothesis of theorem 4.1. Hence ρ is smoothly conjugated to its linear part. Then theorem 5.2 give us that the conjugacy h splits w.r.t. the coarse Lyapunov splitting. Finally, corollary 5.7 and lemma 5.8 give us the theorem. □

Proof. **of Main Theorem 2.1** Take now a linear action $\rho_* : \mathbb{Z}^k \to \mathrm{GL}(N, \mathbb{Z})$ as in theorem 2.1 and an Anosov action ρ whose action in homology is ρ_*. As we said we have a Hölder continuous conjugacy h such that $h \circ \rho = \rho_* \circ h$ and we want to prove that it is smooth. Let n_0 be such that $f = \rho(n_0)$ is an Anosov diffeomorphism, let $A = \rho_*(n_0)$ and C be the ρ_*-Weyl chamber such that $n_0 \in C$. Let p be a periodic point for f and Γ_p the stabilizer of p, that is, the finite index subgroup of $\mathbb{Z}^k$ that leave p fixed. Let $W^s(p)$ be the stable manifold of p for f, we can identify it with $\mathbb{R}^d$ and we have that the restriction of ρ to Γ_p leave $W^s(p)$ invariant. We have that $h(p)$ is a periodic point for ρ_* and that $\Gamma_p = \Gamma_{h(p)}$ so we can work also with the restriction of ρ_* to Γ_p. The stable space of $h(p)$ for A, $h(p) + E^s_C$ can also be identified with $\mathbb{R}^d$. Hence we can work with the actions ρ and ρ_* induced on this $\mathbb{R}^d$ and fixing the origin. They are Hölder conjugated by the restriction of h to these stable manifolds. We can see that hypothesis i), ii) and iii) in theorem 2.1 guaranty that we are in the hypothesis of theorem 6.1. So we have that h restricted to the stable manifold of p is a diffeomorphism outside the complementary Lyapunov spaces. Let us see that it is in fact absolutely continuous and hence that the Lyapunov exponent at p of the restriction of ρ to $W^s(p)$ coincide with the Lyapunov exponents of the restriction of ρ_* to E^s_C.

Let us work in the universal covering $\mathbb{R}^N$. We shall use the same notation for the objects in the torus or in $\mathbb{R}^N$ whenever this leads not to confusion. It is not hard to see that there is $n \in \mathbb{Z}^N$ such that $E^u_C \cap (L + n) = \emptyset$ for every complementary Lyapunov space $L \subset E^s_C$. Let us define the holonomy map $\pi^*_n : h(p) + E^s_C \to h(p) + n + E^s_C$ sliding along E^u_C, that is $\pi^*_n(y) = (h(p) + n + E^s_C) \cap (y + E^u_C)$, π^*_n is an affine map and hence it is smooth. The choice of n implies that $\pi^*_n(h(p)) \notin L$ for every complementary Lyapunov space $L \subset E^s_C$. We have also the unstable holonomy for f, $\pi_n : W^s(p) \to W^s(p + n) = W^s(p) + n$ defined by $\pi_n(x) = (W^s(p) + n) \cap W^u(x)$. Since f is $C^{1+\text{Hölder}}$ it follows that π_n is absolutely continuous with nonzero jacobian. On the other hand, as the conjugacy send the stable and unstable foliations

into the corresponding ones and it is homotopic to the identity we have that it conjugates the unstable holonomies, that is $h \circ \pi_n = \pi_n^* \circ h$ and hence $\pi_n(p)$ is not in the preimage of the complementary Lyapunov spaces. So we have that $h = (\pi_n^*)^{-1} \circ h \circ \pi_n$ and hence h is written in a neighborhood of p as the composition of π_n that is absolutely continuous, h restricted to a neighborhood of $\pi_n(p)$ that is a diffeomorphism by the choice of n and the inverse of π_n^* that is also a diffeomorphism so we have that in a neighborhood of p, h is the composition of π_n with a diffeomorphism and hence it is absolutely continuous in a neighborhood of p and by conclusion b) of theorem 6.1 we have that h is absolutely continuous when restricted to $W^s(p)$ and the Lyapunov exponents for ρ restricted to $W^s(p)$ at p coincide with the ones of ρ^* restricted to E^s_C. Working with f^{-1} we get the same for the unstable manifold and as p was arbitrary we have that the Lyapunov exponents at any point for ρ coincide with the ones of ρ_*.

On the other hand, using corollary 3.6 we get that $\rho(n)$ is Anosov for any element $n \in C$, the ρ^*-Weyl chamber containing n_0. Take an element $n_1 \in C$ satisfying the hypothesis iv) of theorem 2.1, by corollary 3.10 we have that the stable foliation is smooth, similarly working with f^{-1} and $-n_0 \in -C$ we get that the unstable foliation is smooth. Hence the holonomy π_n is smooth and hence by the same argument we used to prove that h was absolutely continuous but now using conclusion c) of theorem 6.1 we get that h restricted to $W^s(p)$ is smooth at p and hence $h|W^s(p)$ is a diffeomorphism. Similarly $h|W^u(p)$ is a diffeomorphism. Finally as the stable and unstable foliations are smooth we get that h is a diffeomorphism and we are done. $\square$

We want to mention also that from a careful reading of the proof, it can be seen that the C^1 distance of the conjugacy to the identity depends only on the C^2 distance of the Anosov element of the action to the linear one and some properties on the linear action. In fact, this estimate comes from theorem 4.1 and from the bounds on the regularity of the invariant foliations that are controlled since the eigenvalues of the Anosov element coincide with the linear ones.

References

[1] M.I. Brin, *Nonwandering points of Anosov diffeomorphisms.* Astérisque **49** (1977), 11–18.

[Ca] Y. Cao, *Non-zero Lyapunov exponents and uniform hyperbolicity.* Nonlinearity 16 (2003) 1473-1479.

[2] J. Franks, *Anosov diffeomorphisms.* 1970 Global Analysis Proc. Sympos. Pure Math, Vol XIV, Berkeley, Calif. (1968), 61–93.

[3] H. Furstenberg, *Disjointness in ergodic theory, minimal sets, and a problem in diophantine analysis.* Math. Syst. Theory, **1** (1967), 1–49.

[4] P. Hartman, *On local homeomorphisms of Euclidean subspaces.* Bol. Soc. Mat. Mexicana **5** (1960), 220–241.

[5] M. Hirsch, C. Pugh and M. Shub, *Invariant manifolds.* Lecture Notes in Mathematics, Vol. 583. Springer-Verlag, Berlin-New York, 1977.

[6] A. Johnson, *Measures on the circle invariant under multiplication by a nonlacunary subsemi- group of the integers.* Israel J. of Math. **77** (1992), 211–240.

[7] B. Kalinin and A. Katok, *Invariant measures for actions of higher rank abelian groups.* Proc. Symp. Pure Math, **69**, (2001), 593–637.

[8] B. Kalinin and A. Katok, *Measure rigidity beyond uniform hyperbolicity: Invariant Measures for Cartan actions on Tori.* Preprint (2006).

[9] B. Kalinin, A. Katok and F. Rodriguez Hertz, *Nonuniform measure rigidity.* in preparation.

[10] B. Kalinin and R. Spatzier, *On the classification of Cartan actions.* to appear in GAFA, Geometric And Functional Analysis.

[11] A. Katok, *Lyapunov exponents, entropy and periodic orbits for diffeomorphisms.* Inst. Hautes tudes Sci. Publ. Math. No. **51** (1980), 137–173.

[12] A. B. Katok and J. W. Lewis, *Local rigidity for certain groups of toral automorphism.* Isr. J. of Math. **75** (1991), 203–241.

[13] A. B. Katok, J. W. Lewis and R. J. Zimmer, *Cocycle superrigidity and rigidity for lattice actions on tori.* Topology **35** (1996), 27–38.

[14] A. Katok and F. Rodriguez Hertz, *Uniquness of large invariant measures for $\mathbb{Z}^k$ actions with Cartan homotopy data.* Preprint (2006).

[15] A. Katok and R. Spatzier, *Differential rigidity of Anosov actions of higher rank abelian groups and algebraic lattice actions.* Tr. Mat. Inst. Steklova **216** (1997), Din. Sist. i Smezhnye Vopr., 292319; translation in Proc. Steklov Inst. Math. 1997, no. 1 **216**, 287–314

[16] A. Katok and R. J. Spatzier, *Invariant Measures for Higher Rank Hyperbolic Abelian Actions.* Ergod. Th. & Dynam. Syst. **16** (1996), 751-778.

[17] A. Manning, *There are no new Anosov diffeomorphisms on tori.* Amer. J. Math. **96** (1974), 422–429.

[18] S. Newhouse, *On codimension one Anosov diffeomorphisms.* Amer. J. Math. **92** (1970), 761–770.

[19] C. Pugh and M. Shub, *Ergodic attractors.* Trans. Amer. Math. Soc. **312** (1989), no. 1, 1–54.

[20] C. Pugh, M. Shub and A. Wilkinson, *Hölder Foliations.* Duke Math. J. **86** (1997), no. 3, 517–546.

[21] F. Rodriguez Hertz, *Global rigidity of $\mathbb{Z}^2$ Cartan Actions on $\mathbb{T}^3$.* Preprint (2001).

[22] D. Rudolph, $\times 2$ *and* $\times 3$ *invariant measures and entropy.* Ergod. Th. & Dynam. Syst. **10** (1990), 395–406.

[Sc] S.J. Schreiber, *On growth rates of sub-additive functions for semi-flows.* J. Differential Equations 148 (1998) 334-350.

[23] W. Sun and Z. Wang, *Lyapunov exponents of hyperbolic measures and hyperbolic periodic orbits.* Preprint (2005)

IMERL, Montevideo, Uruguay
E-mail address: frhertz@fing.edu.uy